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Secular Stability of Rapidly Rotating Stars of Arbitrary Structure

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Abstract: This report utilizes Poincaré's criteria to investigate the secular stability of self-gravitating configurations of arbitrary structure in the state of rapid rotation. The potential energy, a knowledge of which is necessary for application of these criteria, has been determined by an extension of Clairaut's method, and its evaluation in terms of suitably chosen generalized coordinates carried out explicitly to quantities of fourth order in superficial oblateness, for configurations of arbitrary internal structure.

The method employed can, moreover, clearly be extended to attain accuracy of any order, at the expense of mere manipulative work which lends itself to machine automation. The angular velocity of axial rotation can be an arbitrary function of position as well as of the time. An application of our results to homogeneous configurations in rigid-body rotation was undertaken to demonstrate that our method, when applied to a case for which a closed solution exists, leads to results which are consistent with it.

I. INTRODUCTION

An investigation of the stability of self-gravitating configurations constructed to represent the models of celestial bodies constitutes an important aspect of current astrophysical engineering, and the aim of its outcome is to reassure that such configurations as we may have in mind can actually exist in the sky as observable objects capable of retaining the observed characteristics for a suitable length of time. As regards its length, there are, in principle, three time scales of very different duration with which we are concerned in the astrophysics of the celestial objects in the stellar or planetary mass range, customarily referred to as *nuclear, contractional* and *vibrational* scales. The former refers to a time span in which a star of given mass can consume its available nuclear fuel. The contractional (or Kelvin) scale measures the time which an astronomical body of given mass will take to contract gravitationally from a state of infinite distension to its present size, whereas the vibrational time scale refers to a period of the fundamental vibration with which the configuration in question can respond to small arbitrary harmonic disturbances.

Of the three, the nuclear time scale is well known to be by far the longest, and no star can be regarded as stable over the respective time span, for its terminal physical characteristics are bound to differ greatly from the initial ones. Therefore, the question of nuclear stability in this sense does not really arise and will be of no concern to us in this report. The contractional time scale τ_c is defined typically by the expression

$$\tau_c = \frac{Gm^2}{RL}\text{s}, \tag{1.1}$$

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where

m denotes the star's mass
R, its radius
L, its energy loss, per unit time, in erg/sec
G . the constant of gravitation.

The typical value of τ_c is likely to be shorter by several orders of magnitude than the nuclear lifetime of a star, and every configuration which retains its internal thermal energy for time intervals of the order of τ_c can be said to be secularly stable if its potential energy in that stage consistently retains a minimum value—which any accidental departures from eqiulibrium could only increase. Lastly, the vibrational time scale τ_ν, represented typically by the ratio of the potential energy W to the moment of inertia I about the center of mass of the respective configuration, can be estimated from the equation

$$\tau_\nu = \left(\frac{W}{I}\right)^{-1/2} = \left(\frac{R^3}{Gm}\right)^{1/2} \mathrm{s}, \tag{1.2}$$

and turns out to be shorter than τ_c by many orders of magnitude. Stability on this time scale (dynamical stability) will obtain if the motion actuated by an arbitrary disturbance is periodic and does not increase (exponentially or otherwise) with time.

Of the two types of stability—secular and dynamical—the former is by far the more important, both from the mathematical and the physical point of view. Dynamical (or ordinary) instability would imply that a configuration so afflicted must exhibit symptoms of its distress on a time scale comparable with τ_ν; i.e., to constitute a *transient* phenomenon lasting from days to weeks (an interval certainly short in comparison with the time span of our observations). Configurations which do not exhibit transient phenomena on this scale must, therefore, be ordinarily stable; but whether or not they are also secularly stable cannot obviously be disclosed by observations carried out within time spans comparable with a human life. The answer is, however, all-important for the interpretation of our momentary glimpse of the evolutionary stages of the stars in the Universe around us; for configurations which are secularly unstable should obviously be very much less frequently encountered among them than those which are secularly stable.

Unlike the question of ordinary stability of the stars which can often be answered by the observer within a reasonable time span, the existence (or otherwise) of the secular stability can be established only by the mathematician. Secular stability is, moreover, intrinsically by far the more important of the two, for it can be shown (Ref. 1, p. 22) that if a self-gravitating configuration in equilibrium is secularly stable, it must be also ordinarily stable. On the other hand, a configuration ordinarily stable may be secularly stable or unstable; and the same is true of ordinarily unstable configurations as well (though the latter case is of no astronomical interest except in connection with manifestly transient phenomena). Thus while ordinary stability cannot tell us anything definite about the long-range prospects of the star's behavior, the secular stability permits us to draw definite conclusions on the ordinary stability as well; and this underlines the importance of its study.

In the next section following these introductory remarks, the mathematical criteria of secular stability à la Dirichlet and Poincaré will be set up and used in subsequent parts of the report to study the extent of secular stability of rapidly rotating stellar configurations of arbitrary structure. In their classical form (to which we shall adhere throughout this paper) these criteria depend solely on the behavior of the potential energy of the respective configuration; the mathematical crux of the problem will be to evaluate this energy as a mass integral of the potential function. This will be done explicitly to quantities of fourth order in superficial oblateness and compared with the results of less general theories valid for a particular model.

II. CONDITIONS OF SECULAR STABILITY

In what follows, let T denote the kinetic energy of a self-gravitating configuration relative to its center of mass, and W be the potential energy arising from the distribution of mass in the interior (representing the amount of work done by the contraction of our body from infinity to its present state). If so, then (as is well known) the Lagrangian equations governing any motion of our configuration are of the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial W}{\partial q_i} = 0, \qquad i = 1, 2, 3, \ldots, \tag{2.1}$$

where the q_i's represent the generalized coordinates (parameters) on which T and W depend. In the state of *equilibrium,* in which there is no motion, the respective configurations are specified from Eq. (2.1) by the equations

$$\frac{\partial W}{\partial q_i} = 0, \qquad i = 1, 2, 3, \ldots, \tag{2.2}$$

which represent the conditions that W be *stationary* for a certain set of generalized coordinates $q_i \equiv a_i$ obtained by the solution of Eq. (2.2).

In the neighborhood of such a set of parameters describing the state of equilibrium, let the generalized coordinates

$$q_i = a_i + \delta q_i. \tag{2.3}$$

If, moreover, the potential energy $W(q_i)$ represents a function which is analytic for $q_i = a_i$, it should admit, at that point, of a Taylor expansion of the form

$$W(q_i) = W(a_i) + \left(\frac{\partial W}{\partial q_i}\right)_{a_i} \delta q_i + \frac{1}{2}\left(\frac{\partial^2 W}{\partial q_r \partial q_s}\right)_{a_r a_s} \delta q_r \delta q_s + \ldots, \tag{2.4}$$

in which, for sufficiently small δq_i's, the remainder of the series on the r.h.s. of Eq. (2.4) can be ignored. Moreover, in the state of equilibrium,

$$\left(\frac{\partial W}{\partial q_i}\right)_{a_i} = 0 \tag{2.5}$$

by Eq. (2.2); so that, in the neighborhood of equilibrium,

$$W(q_i) = W(a_i) + \frac{1}{2} W_{rs} \delta q_r \delta q_s + \ldots, \tag{2.6}$$

where W_{rs} will hereafter denote the partial derivatives of W with respect to q_r and q_s at $q_i = a_i$

Now, in accordance with a theorem first enunciated by Lagrange and rigorously proved by Dirichlet (2), *the necessary and sufficient condition for the equilibrium configuration to be secularly stable is that $W(a_i)$ represents an absolute minimum of $W(q_i)$.* This will be the case provided that the quadratic function

$$\delta W = \frac{1}{2} W_{rs} \delta q_r \delta q_s \tag{2.7}$$

be *positive definite* for all values of the variables δq_r and δq_s.*

To establish the conditions for which the homogeneous quadratic function (2.7) remains positive, let us introduce a linear transformation of variables

$$\begin{aligned} \delta q_1 &= c_{11}\theta_1 + c_{12}\theta_2 + \ldots + c_{1n}\theta_n , \\ \delta q_2 &= c_{21}\theta_1 + c_{22}\theta_2 + \ldots + c_{2n}\theta_n , \\ &\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ \delta q_n &= c_{n1}\theta_1 + c_{n2}\theta_2 + \ldots + c_{nn}\theta_n , \end{aligned} \tag{2.8}$$

such that the determinant

$$\|c_{ij}\| \equiv \lambda > 0; \tag{2.9}$$

and determine the c_{ij}'s in such a way that the r.h.s. of the quadratic function (2.7) will reduce to a sum of the squares of the form

$$\delta W = \frac{1}{2} (b_1\theta_1^2 + b_2\theta_2^2 + \ldots + b_n\theta_n^2), \tag{2.10}$$

where the coefficients of stability b_i (Ref. 3) are given by the determinantal equation of the form

$$\begin{vmatrix} b_1 & 0 & \ldots & 0 \\ 0 & b_2 & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & b_j \end{vmatrix} = \lambda^2 \begin{vmatrix} W_{11} & W_{12} & \ldots & W_{ij} \\ W_{21} & W_{22} & \ldots & W_{2j} \\ \vdots & & & \vdots \\ W_{j1} & W_{j2} & \ldots & W_{jj} \end{vmatrix} \tag{2.11}$$

or, in brief,

$$b_1 b_2 \ldots b_j = \lambda^2 \Delta_j \qquad \text{for } j = 1, 2, 3, \ldots, n, \tag{2.12}$$

in which $W_{ij} \equiv W_{ji}$, rendering the Hessian determinant Δ_n of W on the r.h.s. of Eq. (2.11) diagonally symmetrical.

The coefficients b_j are by no means defined by Eq. (2.11) uniquely, for in Eq. (2.8) we introduced n^2 arbitrary constants c_{ij} to remove $n(n-1)/2$ cross-products $\delta q_r \delta q_s$ from the quadratic function (2.7); and since $n^2 > n(n-1)/2$, this can be accomplished in an infinity of ways. However, whichever particular transformation we employ, the number of the coefficients b_i which are positive or negative are in each case the same (3), and it is always possible to choose the c_{ij}'s in such a way that $\|c_{ij}\| \equiv \lambda = 1$, a convention which we shall hereafter adopt.

*Should, perchance, all W_{rs} be identically zero, the existence of a minimum would require the positivity of the first group of the nonvanishing partial derivatives of W of lowest *even* order on the r.h.s. of Eq. (2.4).

Let us, however, return to Eq. (2.10). Since the squares θ_i^2 of all new coordinates on its r.h.s. are necessarily positive, the quadratic function δW as defined by Eq. (2.10) will be positive definite if all the coefficients b_i os stability are also positive—i.e., if all minors Δ_j of the Hessian on the r.h.s. of Eq. (2.11) are positive for $j = 1, 2, 3, \ldots, n$. This will be true if

$$b_1 = W_{11} > 0, \tag{2.13}$$

$$b_1 b_2 = \begin{vmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{vmatrix} > 0, \tag{2.14}$$

$$b_1 b_2 b_3 = \begin{vmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{vmatrix} > 0, \tag{2.15}$$

$$b_1 b_2 b_3 \ldots b_n = \begin{vmatrix} W_{11} & W_{12} & \ldots & W_{1n} \\ W_{21} & W_{22} & \ldots & W_{2n} \\ \vdots & \vdots & & \vdots \\ W_{n1} & W_{n2} & \ldots & W_{nn} \end{vmatrix} > 0. \tag{2.16}$$

Therefore, a sufficient condition* for the stability of our configuration is the positivity of all the b_j's. The vanishing of any one of them (which would precede a change of its sign) can occur only if the respective $\Delta_j = 0$, a point at which stability may be lost (or gained). Moreover, any b_i which offends stability by its sign can clearly be removed by a removal of the requisite degree of freedom. A restriction in the degrees of freedom may indeed restore stability, though an increase in their number will never do so.

Thus far we have considered the criteria for stability of a stationary configuration in hydrostatic equilibrium and did not take account of the fact that such a configuration may be endowed by axial rotation. To generalize our stability criteria for such a case, in what follows, let H and T denote the angular momentum and kinetic energy relative to a fixed (inertial) frame of reference, while H_R and T_R represent the same quantity relative to a frame rotating with an angular velocity ω about one (say, the z-) axis. Moreover, let C be the moment of inertia about this axis, and

$$\begin{aligned} u &= \dot{x} - y\omega, \\ v &= \dot{y} + x\omega, \\ w &= \dot{z}, \end{aligned} \tag{2.17}$$

be the rectangular velocity components relative to the inertial frame of reference.

*This condition is sufficient, but *not* necessary for our purpose; for even should one (or more) of the b_j's become negative, it may still be possible to maintain the positivity of δW (the necessary and sufficient condition for stability) by restricting the range of the θ_i^2 which factor the negative b_i's.

If so, then the total kinetic energy T can be expressed in the form of the following three mass integrals:

$$\begin{aligned} T &= \frac{1}{2} \int (u^2 + v^2 + w^2)\, dm \\ &= \frac{1}{2} \int (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\, dm + \omega \int (x\dot{y} - y\dot{x})\, dm + \frac{1}{2} \omega^2 \int (x^2 + y^2)\, dm \qquad (2.18) \\ &= T_R + \omega H_R + \frac{1}{2} \omega^2 C; \end{aligned}$$

and, similarly, the angular momentum

$$H = H_R + \omega C. \qquad (2.19)$$

If we temporarily disregard possible internal motions in our body, and set the *body* velocity components with respect to the *space* axes equal to zero,

$$\dot{x} = \dot{y} = \dot{z} = 0 \qquad (2.20)$$

on the r.h.s. of Eqs. (2.18), and the expressions for T and H as given by Eqs. (2.18) and (2.19) will reduce to

$$T = \frac{1}{2} \omega^2 C \quad \text{and} \quad H = \omega C, \qquad (2.21)$$

respectively.

On the other hand, the Lagrangian equations of motion (2.1) are known to admit of the integral

$$T + W = \text{constant}, \qquad (2.22)$$

which, on insertion for T from Eqs. (2.21) yields

$$W + \frac{1}{2} \omega^2 C = \text{constant}. \qquad (2.23)$$

In the state of equilibrium,

$$\delta \left(W + \frac{1}{2} \omega^2 C \right) = 0, \qquad (2.24)$$

which yields

$$\delta W = -\frac{1}{2} \delta (\omega^2 C). \qquad (2.25)$$

Therefore, the potential energy to be minimized for the rotating configurations becomes (for constant ω) equal to

$$W + \delta W = W - \frac{1}{2} \omega^2 C, \qquad (2.26)$$

and this expression should replace W in all developments in the preceding Eqs. (2.1)-(2.16) of this section.

The minimization of Eq. (2.26) should result in a linear series of equilibrium configurations in which the angular velocity ω is allowed to vary sufficiently slowly for the velocity components of such motion to remain negligible, and the configuration retain its equilibrium state. If, however, we restrict such variations to those in which the angular momentum H remains secularly constant, then on elimination of ω between Eqs. (2.21) the kinetic energy

$$T = \frac{H^2}{2C}, \tag{2.27}$$

which on insertion into the energy integral of Eq. (2.22) requires that, in this case, the expression to be minimized will assume the form

$$W + \frac{H^2}{2C} = \text{constant}. \tag{2.28}$$

The corresponding series of configurations for which this is true will be stable provided that this expression is minimized for constant values of H.

Since, under these conditions,

$$\begin{aligned} \delta\left(W + \frac{H^2}{2C}\right) &= \delta W - (H^2/2C^2)\,\delta C \\ &= \delta W - \frac{1}{2}\,\omega^2 \delta C \\ &= \delta\left(W - \frac{1}{2}\,\omega^2 C\right) - \omega^2 \delta C, \end{aligned} \tag{2.29}$$

by $H = \omega C$ it follows that the two stability criteria differ only in amounts of the order of $\omega^2 \delta C$.

The stability criterion represented by the minimization of the function of Eq. (2.26) is classical and has been used in astronomical literature since the days of Laplace; in particular, it underlies all subsequent work by Jacobi and Poincaré. That represented by Eq. (2.28) is of more recent date, and appears to have been first introduced by Schwarzschild (4). A fuller discussion of various aspects of these criteria can be found in Jeans (5) or a recent treatise by Lyttleton (1).

III. DETERMINATION OF THE POTENTIAL

In the preceding section the criteria for the secular stability of self-gravitating equilibrium configurations have been set up in terms of the potential energy W and moment of inertia C of the respective configuration. An application of such criteria requires, however, an explicit knowledge of the second partial derivatives of these quantities with respect to the generalized coordinates q_i of our problem, and in order to ascertain these, a knowledge of both W and C as functions of q_i is an obvious prerequisite. The aim of this section will be to develop an analytical method by which the explicit forms of $W(q_i)$ as well as $C(q_i)$ could be established for configurations of any structure, and to an arbitrary degree of accuracy.

In embarking on this task we wish to note that, like the kinetic energy T or the angular momentum H of the preceding section, the potential energy W and the moment of inertia C can be defined as the mass integrals

$$W = -\frac{1}{2}\int_0^{m_1} \Omega dm \tag{3.1}$$

and

$$C = \int_0^{m_1} (x^2 + y^2)\, dm \tag{3.2}$$

extended over the entire mass m_1 of our configuration (which remains constant in time), where Ω denotes the potential function arising from the particular distribution of this mass throughout the interior.

In the preceding section the internal structure of our configuration did not yet make an explicit appearance, nor did its shape enter except through the limits of integration. A detailed consideration of such properties can, however, no longer be deferred at this stage. In embarking on this task we shall first confine our attention to a determination of the potential Ω arising from the mass, which offers a clue to the potential energy W as defined by Eq. (3.1); for, once the problem of its determination has been solved, the evaluation of C from Eq. (3.2) will constitute a relatively simple task.

To establish the potential of a deformable configuration distorted by rotation or any other force, let us depart from the Eulerian equations of motion of the form

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \operatorname{grad} \Psi - \operatorname{grad} P, \tag{3.3}$$

where ρ denotes the density at any internal point of our configuration and P the corresponding pressure. The quantity

$$\Psi = \Omega + V' \tag{3.4}$$

denotes the total potential—gravitational (Ω) plus disturbing (V')—of all forces acting upon our body; $\mathbf{u}$, the (vector) velocity of any displacement; and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \operatorname{grad}, \tag{3.5}$$

the Lagrangian derivative with respect to the time t. If, moreover, consistent with Eqs. (2.2) our configuration is in equilibrium, all three components of the vector $\mathbf{u}$ vanish identically, and the equilibrium form $\operatorname{grad} P = \rho \operatorname{grad} \Psi$ of Eqs. (3.3) reduces to the three scalar equations

$$\frac{\partial P}{\partial x} = \rho \frac{\partial \Psi}{\partial x}, \quad \frac{\partial P}{\partial y} = \rho \frac{\partial \Psi}{\partial y}, \quad \frac{\partial P}{\partial z} = \rho \frac{\partial \Psi}{\partial z}. \tag{3.6}$$

On equating the mixed second derivatives $\partial^2 P/\partial x \partial y$, $\partial^2 P/\partial x \partial z$ and $\partial^2 P/\partial y \partial z$ obtained by appropriate differentiation of Eqs. (3.6), we find that

$$\frac{\partial\rho}{\partial x}\frac{\partial\Psi}{\partial y} = \frac{\partial\rho}{\partial y}\frac{\partial\Psi}{\partial x}, \quad \frac{\partial\rho}{\partial x}\frac{\partial\Psi}{\partial z} = \frac{\partial\rho}{\partial z}\frac{\partial\Psi}{\partial x}, \quad \frac{\partial\rho}{\partial y}\frac{\partial\Psi}{\partial y} = \frac{\partial\rho}{\partial z}\frac{\partial\Psi}{\partial y}, \tag{3.7}$$

from which it follows that

$$\frac{\dfrac{\partial\rho}{\partial x}}{\dfrac{\partial\Psi}{\partial x}} = \frac{\dfrac{\partial\rho}{\partial y}}{\dfrac{\partial\Psi}{\partial y}} = \frac{\dfrac{\partial\rho}{\partial z}}{\dfrac{\partial\Psi}{\partial z}}. \tag{3.8}$$

The structure of these equations discloses at once that the surfaces ρ = constant must necessarily coincide with those of the equipotentials Ψ = constant. Moreover, the Eqs. (3.5) can, under these circumstances, be rewritten in the form of a single total differential equation

$$dP = \rho d\Psi. \tag{3.9}$$

If Ψ is a function of ρ only, so must (by Eq. (3.9)) be P, this leads to an equation of state of the form

$$P = f(\rho), \tag{3.10}$$

where $f(\rho)$ stands for an arbitrary function of the density ρ. Any surface over which P and ρ are constant must, therefore, be an equipotential;

$$\Psi = \text{constant}. \tag{3.11}$$

The latter equation implies, in fact, a complete specification of our problem; and our task will, in what follows, reduce to detail its explicit form for any given type of disturbing force.

In order to do so, let us fix our attention to an arbitrary point $M(r, \theta, \phi)$ in the interior of our configuration (see Fig. 1) comprised between the radii $r = r_0$ and r_1, and let $M'(r', \theta', \phi')$ be an arbitrary point of this stratum. If so, the *interior potential U* at M will evidently be given by the integral

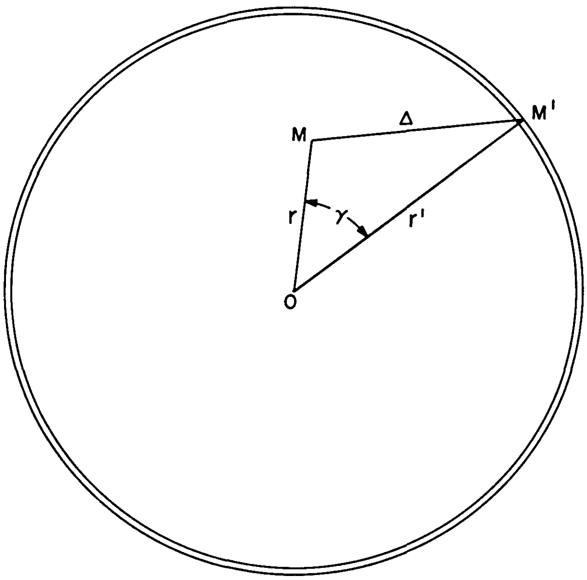

Fig. 1 – Geometry for the determination of the potential

$$U = G \int_{r_0}^{r_1} \frac{dm'}{\Delta}, \tag{3.12}$$

where G denotes, as before, the gravitation constant; dm', the mass element

$$dm' = \rho r'^2 dr' \sin\theta' \, d\theta' \, d\phi';$$

and from the triangle OMM' (cf. again Fig. 1)

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos\gamma, \tag{3.13}$$

where

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi'). \tag{3.14}$$

Moreover, the *exterior potential* V will be given by an analogous expression of the form

$$V = G \int_0^{r_0} \frac{dm'}{\Delta}, \tag{3.15}$$

similar to Eq. (3.12), in which the limits of integration extend from the origin to r_0; and the sum $U + V = \Omega$ constitutes the total potential of our configuration arising from its mass.

To evaluate the two constituents of Ω, let us expand Δ^{-1} in terms of the Legendre polynomials $P_n(\cos\gamma)$ of ascending integral order n in a well-known series of the form

$$\begin{aligned} \frac{1}{\Delta} &= \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\gamma), \quad r' < r, \\ &= \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r}{r'}\right)^n P_n(\cos\gamma), \quad r' > r, \end{aligned} \tag{3.16}$$

which on insertion in Eqs. (3.12) and (3.15) permits us to express U and V as

$$U = \sum_{n=0}^{\infty} r^n U_n \tag{3.17}$$

and

$$V = \sum_{n=0}^{\infty} r^{-n-1} V_n, \tag{3.18}$$

where

$$U_n = G \int_{r_0}^{r_1} \int_0^{\pi} \int_0^{2\pi} \rho (r')^{1-n} P_n(\cos\gamma) \, dr' \sin\theta' \, d\theta' \, d\phi' \tag{3.19}$$

and

$$V_n = G \int_0^{r_0} \int_0^{\pi} \int_0^{2\pi} \rho (r')^{2+n} P_n(\cos\gamma)\, dr' \sin\theta'\, d\theta'\, d\phi'. \tag{3.20}$$

Now in the foregoing expressions, let

$$r' \equiv r(a, \theta', \phi') \tag{3.21}$$

denote symbolically the parametric equation of an equipotential surface of constant density and pressure. By virtue of the uniqueness of the potential function, only one such surface can pass through any point (i.e., belong to any value of a). Since the density must remain constant over such a surface, it follows that ρ can hereafter be regarded as a function of a *single* variable a, introduced by Eq. (3.21) and representing the mean radius of the respective equipotential. As such, it is bounded so that

$$0 \leqslant a \leqslant a_1, \tag{3.22}$$

where a_1 represents the (smallest) root of the equation

$$\rho(a_1) = 0. \tag{3.23}$$

The fact that ρ can thus be regarded as a function of a single variable suggests that it should be of advantage to change over from r' to a as the new variable of integration on the r.h.s. of Eq. (3.19) and (3.20). Inasmuch as the Jacobian J of the transformation from (r', θ', ϕ') to (a, θ', ϕ') is of the form

$$J = \begin{vmatrix} \frac{\partial r'}{\partial a} & \frac{\partial r'}{\partial \theta'} & \frac{\partial r'}{\partial \phi'} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{\partial r'}{\partial a}, \tag{3.24}$$

a transition from r' to a can be effected simply by setting

$$dr' = \frac{\partial r'}{\partial a}\, da. \tag{3.25}$$

If so, however, then evidently

$$U_n = \frac{G}{2-n} \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \left\{ \int_0^{\pi} \int_0^{2\pi} (r')^{2-n} P_n(\cos\gamma) \sin\theta'\, d\theta'\, d\phi' \right\} da \tag{3.26}$$

for $n \gtrless 2$, and

$$U_2 = G \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a} \left\{ \int_0^{\pi} \int_0^{2\pi} (\log r') P_2(\cos\gamma) \sin\theta'\, d\theta'\, d\phi' \right\} da; \tag{3.27}$$

while, similarly,

$$V_n = \frac{G}{n+3}\int_0^{a_0} \rho \frac{\partial}{\partial a}\left\{\int_0^{\pi}\int_0^{2\pi}(r')^{n+3}P_n(\cos\gamma)\sin\theta'\,d\theta'\,d\phi'\right\}da \tag{3.28}$$

for any value of n.

If $n = 0$,

$$U_0 = 4\pi G\int_{a_0}^{a_1} \rho a\,da \tag{3.29}$$

and

$$V_0 = G\int_0^{a_0} dm' = Gm(a_0), \tag{3.30}$$

where $m(a_0)$ denotes the mass of our configuration interior to a_0.

In order to proceed further, let us assume that the radius vector r' of an equipotential surface can be expanded in a series of the form

$$r' = a\left\{1 + \sum_{j=0}^{\infty} Y_j(a,\theta',\phi')\right\}, \tag{3.31}$$

where the Y_j's stand for solid harmonic functions of the respective coordinates, satisfying the partial differential equation

$$\frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\left(\sin\theta'\frac{\partial Y_j}{\partial\theta'}\right) + \frac{1}{\sin^2\theta'}\frac{\partial^2 Y_j}{\partial\phi'^2} + j(j+1)\,Y_j = 0, \tag{3.32}$$

where j is zero or a positive integer. If the solid harmonics $Y_j(a,\theta',\phi')$ can be factored in the form

$$Y_j(a,\theta',\phi') = f_j(a)\,P_j(\theta',\phi'), \tag{3.33}$$

where the P_j's are the same zonal harmonics which occur on the r.h.s. of Eq. (3.16), it is possible to assert, by a well-known orthogonality theorem, that

$$\begin{aligned}\int_0^{\pi}\int_0^{2\pi} & P_n(\cos\gamma)\,Y_j(a,\theta',\phi')\sin\theta'\,d\theta'\,d\phi' \\ &= 0 \qquad\qquad \text{if } j \neq n, \\ &= \frac{4\pi}{2j+1}\,Y_j(a,\theta,\phi) \quad \text{if } j = n.\end{aligned} \tag{3.34}$$

With the aid of this theorem it should be possible—on insertion for r' from Eq. (3.31)—to rewrite the expansions of Eqs. (3.17) and (3.18) for U and V in the form

$$U = 4\pi G \sum_{j=0}^{\infty} \frac{E_j(a)}{2j + 1} r^j P_j(\theta, \phi) \tag{3.35}$$

and

$$V = 4\pi G \sum_{j=0}^{\infty} \frac{F_j(a)}{2j + 1} r^{-j-1} P_j(\theta, \phi), \tag{3.36}$$

where $E_j(a)$ and $F_j(a)$ are appropriate functions of a only.

An explicit evaluation of these functions will be undertaken in the next section. In this place we wish to note that although the sum $\Omega = U + V$ of the foregoing expansions of Eqs. (3.35) and (3.36) constitutes the complete potential arising from the mass of our configuration, it does not yet represent the total potential Ψ; for, in order to specify the latter we must adjoin to Ω the *disturbing potential* V', whose action will cause this configuration from spherical form. In point of fact, the total potential $\Psi = U + V + V'$ must satisfy the Poisson equation

$$\nabla^2 \Psi + 4\pi G \rho = 2\omega^2, \tag{3.37}$$

of which $\Omega = U + V$ represents the particular integral and V' is the complementary function, the form of which remains yet to be specified.

If our configuration were nonrotating, the disturbing potential V' would represent the solution of the homogeneous equation

$$\nabla^2 V' = 0, \tag{3.38}$$

which is known to be of the form

$$V' = \sum_j \left\{ A_j r^j + B_j r^{-j-1} \right\} P_j(\theta, \phi), \tag{3.39}$$

where A_j and B_j are arbitrary constants. The regularity of V' at the origin requires that $B_j = 0$, and the values of A_j then depend on the nature of the disturbing force. Such will, for instance, be the case for distortion arising from tides.

In what follows we plan, however, to confine our present attention to the case of *rotational distortion* of our configuration, arising from uniform rotation* with constant angular velocity ω about one (say, Z-) axis whose direction is fixed in space. In such a case, the differential equation for V' is of the form

$$\nabla^2 V' = 2\omega^2, \tag{3.40}$$

and its regular solution reduces to

$$V' = \frac{1}{2} \omega^2 r^2 \sin^2\theta = A_2 r^2 \{1 - P_2(\cos\theta)\}, \tag{3.41}$$

*An assumption of constant angular velocity is not essential for the validity of our procedure. The latter would continue to hold good—though in more complicated form—as long as $\omega(a, \theta, \phi)$ remains expansible in terms of spherical harmonics.

where

$$A_2 = \frac{1}{3}\,\omega^2. \tag{3.42}$$

If Eqs. (3.35), (3.36), and (3.41) are combined, the total potential of a self-gravitating configuration whose distortion derives from the centrifugal potential Eq. (3.41), will assume the form

$$\begin{aligned}\Psi(r,\theta,\phi) &= 4\pi G \sum_{j=0}^{\infty} \frac{r^j E_j(a) + r^{-j-1} F_j(a)}{2j+1} P_j(\cos\theta) \\ &\quad + \frac{1}{3}\,\omega^2 r^2 \left\{1 - P_2(\cos\theta)\right\},\end{aligned} \tag{3.43}$$

valid to any arbitrary degree of accuracy. Therefore, over an equipotential surface specified by the radius vector r',

$$\begin{aligned}\Psi(r',\theta,\phi) &= 4\pi G \sum_{j=0}^{\infty} \frac{(r')^j E_j(a) + (r')^{-j-1} F_j(a)}{2j+1} P_j(\cos\theta) \\ &\quad + \frac{1}{3}\,\omega^2 (r')^2 \left\{1 - P_2(\cos\theta)\right\},\end{aligned} \tag{3.44}$$

where r' continues to be given by Eqs. (3.31)-(3.33) and where, accordingly, the last term can be expanded into

$$\begin{aligned}\frac{1}{3}\,\omega^2 (r')^2 \left\{1 - P_2(\cos\theta)\right\} &= \frac{1}{3}\,\omega^2 a^2 (1 - f_2) \\ &\quad - \frac{1}{3}\,\omega^2 \left\{1 - 2f_2 + \frac{6}{7} f_2^2\right\} r'^2 P_2(\cos\theta) \\ &\quad + \frac{\omega^2}{a^2} \left\{\frac{2}{3} f_4 - \frac{18}{35} f_2^2\right\} r'^4 P_4(\cos\theta) + \ldots,\end{aligned} \tag{3.45}$$

which, if we regard ω^2 itself as a small quantity of first order, represents an expansion of the centrifugal potential $V'(r')$ correctly to terms of third order.

Suppose next that we expand the r.h.s. of Eq. (3.44) in a Neumann series of the form

$$\Psi(r',\theta,\phi) = \sum_{j=0}^{\infty} \alpha_j(a) P_j(\theta,\phi), \tag{3.46}$$

with coefficients defined by

$$\alpha_j(a) = \frac{2j+1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \Psi(r',\theta,\phi)\, P_j(\theta,\phi) \sin\theta \, d\theta \, d\phi. \tag{3.47}$$

If now—consistent with Eq. (3.11)—the total potential Ψ is to remain constant over a surface of the form Eq. (3.31), it follows that *all terms on the r.h.s. of Eq. (3.46) factored by $P_j(\theta, \phi)$ for $j > 0$ must necessarily vanish*; and this can be true only if we set

$$\alpha_j(a) = 0, \quad j > 0, \tag{3.48}$$

leaving us with α_0 as the constant value characterizing the respective equipotential of mean radius a.

In other words, n equations of the (3.48) can be set up to specify n functions $f_i(a)$, $j = 1, 2, \ldots, n$ occurring on the r.h.s. of the equation

$$r' = a\left\{1 + \sum_{j=0}^{\infty} f_j(a)\, P_j(\cos\theta)\right\}. \tag{3.49}$$

Moreover, once these have been determined, we are in a position to formulate the function $\alpha_0(a)$ for any particular equipotential and proceed with the evaluation of the potential energy W from Eq. (3.1).

In the next section, we shall proceed to set up a system of differential equations whose particular solutions will furnish the desired amplitudes $f_j(a)$, while the completion of our task of evaluation of the potential energy will be deferred to the last section.

IV. DETERMINATION OF THE SHAPE OF RAPIDLY ROTATING CONFIGURATIONS

The present section establishes the explicit form of the amplitudes $f_j(a)$ in the expansion (3.49) for the shape of an equipotential surface distorted by centrifugal force of a rapidly rotating configuration. If this rotation takes place about the Z-axis whose direction is fixed in space, it follows at once that the expansion on the r.h.s. of Eq. (3.49) can consist of only even-order harmonics, on account of symmetry. Moreover, if the coefficient $f_2(a)$ of the first of them (corresponding to $j = 2$) represents a quantity of first order in superficial distortion, $f_4(a)$ will be of the order of f_2^2 or of second order (5, Sec. III-5), $f_6(a)$ of third order, etc.

Now let us, in what follows, set out to develop a consistent theory of the shape of rapidly rotating configurations which is complete to quantities of *third* order in superficial distortion—a scheme of accuracy in which Eq. (3.49) will be restricted to consist of the terms

$$\begin{aligned} r' &= a\left\{1 + f_0 + f_2 P_2(\cos\theta) + f_4 P_4(\cos\theta) + f_6 P_6(\cos\theta) + \ldots\right\} \\ &= a\{1 + \Sigma\}. \end{aligned} \tag{4.1}$$

Within the scheme of our approximation,

$$(r')^{2-n} = a^{2-n}\left\{1 - (n-2)\,\Sigma + \frac{1}{2}(n-1)(n-2)\,\Sigma^2 - \frac{1}{6}\,n(n-1)(n-2)\,\Sigma^3 + \ldots\right\} \tag{4.2}$$

for $n \gtrless 2$ and, for $n = 2$,

$$\log r' = \log a + \Sigma - \frac{1}{2}\Sigma^2 + \frac{1}{3}\Sigma^3 - \ldots, \tag{4.3}$$

while

$$(r')^{n+3} = a^{n+3}\left\{1 + (n+3)\,\Sigma + \frac{1}{2}(n+3)(n+2)\,\Sigma^2 + \frac{1}{6}(n+3)(n+2)(n+1)\,\Sigma^3 + \ldots\right\} \tag{4.4}$$

for any value of n.

Let us decompose the powers and cross-products of Legendre polynomials occurring in different powers of Σ on the r.h.s. of Eqs. (4.2)-(4.4) into their linear combinations by use of the well-known formula which asserts that, for $m \leqslant n$,

$$P_m P_n = \sum_{j=0}^{m} \frac{A_{m-j}A_j A_{n-j}}{A_{m+n-j}} \left\{\frac{2m + 2n + 1 - 4j}{2m + 2n + 1 - 2j}\right\} P_{m+n-2j}, \tag{4.5}$$

where $A_0 = 1$ and, for $j > 0$,

$$A_j = \frac{1.3.5 \ldots (2j-1)}{j!}. \tag{4.6}$$

The orthogonality properties of the P_n's are such that

$$\begin{aligned} \int_{-1}^{1} P_m P_n d\cos\theta &= 0 \qquad \text{if } m \neq n, \\ &= \frac{2}{2n+1} \qquad \text{if } m = n. \end{aligned} \tag{4.7}$$

If so, then the insertion of (4.1)-(4.7) together with the use of the orthogonality theorem (3.34) in Eqs. (3.26)-(3.28) for U_n and V_n should enable us to express the latter in the forms

$$(2n+1)\,U_n = 4\pi G E_n(a) P_n(\cos\theta), \tag{4.8}$$

$$(2n+1)\,V_n = 4\pi G F_n(a) P_n(\cos\theta), \tag{4.9}$$

the amplitudes of which we shall now proceed to evaluate.

To begin, we note that V_0 as defined by (3.30) can be written as

$$\begin{aligned} F_0 &= \frac{1}{12\pi}\int_0^a \rho \frac{\partial}{\partial a}\left\{\int_0^\pi \int_0^{2\pi} (r')^3 \sin\theta\, d\theta\, d\phi\right\} da \\ &= \int_0^a \rho \frac{\partial}{\partial a}\left\{a^3\left[\frac{1}{3} + (1+f_0)\left(f_0 + \frac{1}{5}f_2^2\right) + \frac{2}{105}f_2^3 + \ldots\right]\right\} da \end{aligned} \tag{4.10}$$

(in which the zero subscript of the upper limit will hereafter be dropped) and represents the mass of our configuration interior to a. Since this mass must obviously be independent of distortion, it follows that the zero-harmonic amplitude f_0 is constrained to satisfy the equation

$$(1+f_0)\left(f_0+\frac{1}{5}f_2^2\right)+\frac{2}{105}f_2^3+\ldots=0, \tag{4.11}$$

which yields

$$f_0=-\frac{1}{5}f_2^2-\frac{2}{105}f_2^3-\ldots, \tag{4.12}$$

correctly to quantities of third order.

Taking advantage of this fact, we establish—after some algebra—that, to the same order of accuracy,

$$E_0=\int_a^{a_1}\rho\,\frac{\partial}{\partial a}\left\{a^2\left[\frac{1}{2}-\frac{1}{10}f_2^2-\frac{2}{105}f_2^3\right]\right\}da, \tag{4.13}$$

$$E_2=\int_a^{a_1}\rho\,\frac{\partial}{\partial a}\left\{f_2-\frac{1}{7}f_2^2+\frac{12}{35}f_2^3-\frac{2}{7}f_2f_4\right\}da, \tag{4.14}$$

$$E_4=\int_a^{a_1}\rho\,\frac{\partial}{\partial a}\left\{\frac{1}{a^2}\left[f_4-\frac{27}{35}f_2^2+\frac{216}{385}f_2^3-\frac{60}{77}f_2f_4\right]\right\}da, \tag{4.15}$$

$$E_6=\int_a^{a_1}\rho\,\frac{\partial}{\partial a}\left\{\frac{1}{a^4}\left[f_6+\frac{90}{77}f_2^3-\frac{25}{11}f_2f_4\right]\right\}da; \tag{4.16}$$

and, similarly,

$$F_0=\int_0^a\rho a^2da, \tag{4.17}$$

$$F_2=\int_0^a\rho\,\frac{\partial}{\partial a}\left\{a^5\left[f_2+\frac{4}{7}f_2^2+\frac{2}{35}f_2^3+\frac{8}{7}f_2f_4\right]\right\}da, \tag{4.18}$$

$$F_4=\int_0^a\rho\,\frac{\partial}{\partial a}\left\{a^7\left[f_4+\frac{54}{35}f_2^2+\frac{108}{77}f_2^3+\frac{120}{77}f_2f_4\right]\right\}da, \tag{4.19}$$

$$F_6=\int_0^a\rho\,\frac{\partial}{\partial a}\left\{a^9\left[f_6+\frac{24}{11}f_2^3+\frac{40}{11}f_2f_4\right]\right\}da. \tag{4.20}$$

As the next step of our procedure, let us establish the explicit form of Eqs. (3.48) for $j = 2$, 4, and 6. On evaluating the integrals on the r.h.s. of Eq. (3.47) by the same method, we find that, for $j = 2, 4, 6$ Eqs. $\alpha_j(a) = 0$ assume the more explicit forms

$$\begin{aligned}
&\frac{a^2E_2}{5}\left\{1+\frac{4}{7}f_2+\frac{4}{7}f_4+\frac{1}{35}f_2^2\right\}+\frac{8}{63}a^4f_2E_4\\
&-\frac{F_0}{a}\left\{f_2-\frac{2}{7}f_2^2+\frac{29}{35}f_2^3-\frac{4}{7}f_2f_4\right\}\\
&+\frac{F_2}{5a^3}\left\{1-\frac{6}{7}f_2-\frac{6}{7}f_4+\frac{111}{35}f_2^2\right\}-\frac{10}{63}\frac{f_2}{a^5}F_4\\
&=\frac{\omega^2a^2}{12\pi G}\left\{1-\frac{10}{7}f_2-\frac{9}{35}f_2^2+\frac{4}{7}f_4\right\};
\end{aligned}\tag{4.21}$$

$$\begin{aligned}
&\frac{a^2E_2}{5}\left\{\frac{36}{35}f_2+\frac{40}{77}f_4+\frac{108}{385}f_2^2\right\}+\frac{a^4E_4}{9}\left\{1+\frac{80}{77}f_2\right\}\\
&-\frac{F_0}{a}\left\{f_4-\frac{18}{35}f_2^2+\frac{108}{385}f_2^3-\frac{40}{77}f_2f_4\right\}\\
&-\frac{F_2}{5a^3}\left\{\frac{54}{35}f_2+\frac{60}{77}f_4-\frac{648}{385}f_2^2\right\}+\frac{F_4}{9a^5}\left\{1-\frac{100}{77}f_2\right\}\\
&=\frac{\omega^2a^2}{6\pi G}\left\{\frac{18}{35}f_2-\frac{57}{77}f_4-\frac{9}{77}f_2^2\right\};
\end{aligned}\tag{4.22}$$

$$\begin{aligned}
&\frac{a^2E_2}{5}\left\{\frac{10}{11}f_4+\frac{18}{77}f_2^2\right\}+\frac{20}{99}a^4f_2E_4+\frac{1}{13}a^6E_6\\
&-\frac{F_0}{a}\left\{f_6+\frac{18}{77}f_2^3-\frac{10}{11}f_2f_4\right\}-\frac{3}{11}\frac{F_2}{a^3}\left\{f_4-\frac{36}{35}f_2^2\right\}\\
&-\frac{25}{99}\frac{f_2F_4}{a^5}+\frac{F_6}{13a^7}\\
&=\frac{\omega^2a^2}{6\pi G}\left\{\frac{5}{11}f_4+\frac{9}{77}f_2^2\right\}.
\end{aligned}\tag{4.23}$$

To reduce these equations to more symmetrical forms, let us note that, to the *first* order in small quantities, Eq. (4.21) reduces to

$$\frac{a^2E_2}{5}+\frac{F_2}{5a^3}-\frac{f_2F_0}{a}=\frac{\omega^2a^2}{12\pi G}\tag{4.24}$$

and, with its aid, Eqs. (4.21) and (4.22) yield

$$\frac{a^2E_2}{5}+\frac{F_2}{5a^3}\left\{1-\frac{10}{7}f_2\right\}-\frac{f_2F_0}{a}\left\{1-\frac{6}{7}f_2\right\}=\frac{\omega^2a^2}{12\pi G}\left\{1-2f_2\right\}\tag{4.25}$$

and

$$\frac{a^4 E_4}{9} + \frac{F_4}{9a^5} - \frac{18}{35}\frac{f_2 F_2}{a^3} - \frac{F_0}{a}\left\{f_4 - \frac{54}{35} f_2^2\right\} = 0, \tag{4.26}$$

correctly to quantities of *second* order. By insertion from the foregoing equations (4.24)-(4.26) in (4.21)-(4.23) for terms which are multiplied by small quantities, it is possible to rewrite (4.21)-(4.23) correctly to terms of third order in the following alternative forms

$$\begin{aligned} \frac{a^2 E_2}{5} + \frac{F_2}{5a^3} - \frac{f_2 F_0}{a} = &-\frac{F_0}{a}\left\{\frac{6}{7} f_2^2 - \frac{748}{245} f_2^3 + \frac{16}{7} f_2 f_4\right\} \\ &+ \frac{F_2}{5a^3}\left\{\frac{10}{7} f_2 - \frac{338}{49} f_2^2 + \frac{10}{7} f_4\right\} \\ &+ \frac{2f_2}{7a^5} F_4 + \frac{a^2}{3}\left(\frac{\omega^2}{4\pi G}\right)\left\{1 - 2f_2 + \frac{6}{7} f_2^2\right\}, \end{aligned} \tag{4.27}$$

$$\begin{aligned} \frac{a^4 E_4}{9} + \frac{F_4}{9a^5} - \frac{f_4 F_0}{a} = &-\frac{F_0}{a}\left\{\frac{54}{35} f_2^2 - \frac{1890}{2695} f_2^3 + \frac{160}{77} f_2 f_4\right\} \\ &+ \frac{F_2}{5a^3}\left\{\frac{18}{7} f_2 - \frac{2988}{539} f_2^2 + \frac{100}{77} f_4\right\} \\ &+ \frac{20 f_2}{77a^5} F_4 - \frac{\omega^2 a^2}{4\pi G}\left\{\frac{2}{3} f_4 - \frac{18}{35} f_2^2\right\}, \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} \frac{a^6 E_6}{13} + \frac{F_6}{13a^7} - \frac{f_6 F_0}{a} = &\frac{F_0}{a}\left\{\frac{216}{77} f_2^3 - \frac{40}{11} f_2 f_4\right\} \\ &+ \frac{F_2}{a^3}\left\{\frac{5}{11} f_4 - \frac{90}{77} f_2^2\right\} + \frac{5f_2}{11a^5} F_4. \end{aligned} \tag{4.29}$$

The same equations would have been obtained had we expanded the r.h.s. of Eq. (3.44) for $\Psi(r', \theta, \phi)$ in terms of the products $r'^j P_j(\cos\theta)$ and equated their coefficients for $j = 2, 4, 6 \ldots$ to zero.

The foregoing Eqs. (4.27)-(4.29) contain the unknown amplitudes f_6 both in front of, and behind, the integral signs in the expressions for E_j and F_j as given by Eqs. (4.13)-(4.20). To lure them out from behind the integral signs, multiply Eqs. (4.27)-(4.29) by a^j ($j = 2, 4, 6$), respectively (so as to render the coefficients of E_j on the left-hand sides constant), and differentiate with respect to a. The derivatives of E_j and F_j are merely equal to the integrands on the r.h.s. of Eqs. (4.13)-(4.20); however, since (unlike for F_j) the independent variable a occurs in the lower limit of the definite integrals for E_j, the derivatives of E_j are equal to the integrands on the r.h.s. of (4.13)-(4.16) taken with the negative sign. If, subsequently, we eliminate the terms F_n for $n \neq j$ factored by small quantities with the aid of Eqs. (4.24)-(4.26) valid to the accuracy of lower orders, it is possible to express the F_j's for $j = 2, 4, 6$ in the form

$$F_2(a) = a^2F_0\left\{3f_2 - af_2' + \frac{12}{7}f_2^2 - \frac{4}{7}f_2(af_2') + \frac{2}{7}(af_2')^2\right.$$

$$- \frac{102}{35}f_2^3 - \frac{2}{7}f_2^2(af_2') - \frac{4}{7}f_2(af_2')^2 - \frac{8}{35}(af_2')^3 \tag{4.30}$$

$$\left. + \frac{52}{7}f_2f_4 - \frac{4}{7}a(f_2f_4' + f_4f_2') + \frac{4}{7}a^2f_2'f_4'\right\}$$

$$+ \frac{2}{3}\left(\frac{\omega^2a^5}{4\pi G}\right)\left\{af_2' + \frac{4}{7}f_2(af_2') - \frac{2}{7}(af_2')^2\right\},$$

$$F_4(a) = a^4F_0\left\{5f_4 - af_4'\right.$$

$$+ \frac{108}{35}f_2^2 - \frac{72}{35}f_2(af_2') + \frac{18}{35}(af_2')^2$$

$$+ \frac{972}{385}f_2^3 - \frac{612}{385}f_2^2(af_2') + \frac{324}{385}f_2(af_2')^2 - \frac{108}{385}(af_2')^3 \tag{4.31}$$

$$\left. + \frac{520}{77}f_2f_4 - \frac{80}{77}a(f_2f_4' + f_4f_2') + \frac{40}{77}a^2f_2'f_4'\right\}$$

$$+ 2\left(\frac{\omega^2a^7}{4\pi G}\right)\left\{-\frac{2}{3}f_4 + \frac{1}{3}af_4' + \frac{18}{35}f_2^2 + \frac{24}{35}f_2(af_2') - \frac{6}{35}(af_2')^2\right\},$$

and

$$F_6(a) = a^6F_0\left\{7f_6 - af_6' + \frac{270}{77}f_2^3 - \frac{18}{7}f_2^2(af_2')\right.$$

$$+ \frac{90}{77}f_2(af_2')^2 - \frac{18}{77}(af_2')^3 + \frac{130}{11}f_2f_4 \tag{4.32}$$

$$\left. - \frac{30}{11}a(f_2f_4' + f_4f_2') + \frac{10}{11}a^2f_2'f_4'\right\},$$

where primes denote differentiation with respect to a, correctly to terms of third order.

As the last step of our analysis, let us differentiate the foregoing Eqs. (4.30)-(4.32) once more with respect to a, and insert for $F_j'(a)$ from (4.17)-(4.20); the outcome discloses that the amplitudes $f_j(a)$ for $j = 2, 4, 6$ should satisfy the following second-order differential equations:

$$
\begin{aligned}
a^2 f_2'' + 6D(af_2' + f_2) - 6f_2 = {} & \frac{2}{7}\Big\{2\eta_2(\eta_2+9) - 9D\eta_2(\eta_2+2)\Big\} f_2^2 \\
& - \frac{4}{35}\Big\{(7\eta_2^3 + 33\eta_2^2 + 180\eta_2 + 66) \\
& + 3D\left(2\eta_2^3 - 15\eta_2^2 - 27\eta_2 + 5\right)\Big\} f_2^3 \\
& + \frac{4}{7}\Big\{2(\eta_2\eta_4 + 15\eta_2 + 8\eta_4) \\
& - 3D(3\eta_2\eta_4 + 3\eta_2 + 3\eta_4 - 7)\Big\} f_2 f_4 \\
& + \frac{3\omega^2}{\pi G\bar{\rho}}(1-D)\Big\{f_2 + af_2' + \frac{6}{7} f_2(af_2') + \frac{3}{7}(af_2')^2\Big\} \\
& + \frac{1}{6}\left(\frac{3\omega^2}{\pi G\bar{\rho}}\right)^2 (1-D)(\eta_2+1) f_2,
\end{aligned}
\tag{4.33}
$$

$$
\begin{aligned}
a^2 f_4'' + 6D(af_4' + f_4) - 20f_4 = {} & \frac{18}{35}\Big\{2\eta_2(\eta_2+2) - 3D(3\eta_2^2 + 6\eta_2 + 7)\Big\} f_2^2 \\
& + \frac{36}{385}\Big\{2(3-\eta_2)(1-5\eta_2) \\
& - 3D\left(3\eta_2^3 + 9\eta_2^2 + 12\eta_2 + 4\right)\Big\} f_2^3 \\
& + \frac{40}{77}\Big\{(2\eta_2\eta_4 + 23\eta_2 + 9\eta_4) - 9D(\eta_2\eta_4 + \eta_2 + \eta_4)\Big\} f_2 f_4 \\
& + \frac{3\omega^2}{\pi G\bar{\rho}}(1-D)\Big\{f_4 + af_4' + \frac{9}{35}\Big[7f_2^2 + 6f_2(af_2') + 3(af_2')^2\Big]\Big\},
\end{aligned}
\tag{4.34}
$$

and

$$
\begin{aligned}
& a^2 f_6'' + 6D(af_6' + f_6) - 42f_6 \\
& = \frac{18}{77}\Big\{4(3-\eta_2)(\eta_2+2) - 3D(\eta_2^3 + 3\eta_2^2 + 15\eta_2 + 5)\Big\} f_2^3 \\
& + \frac{10}{11}\Big\{2(\eta_2\eta_4 + 6\eta_2 - \eta_4) - 3D(3\eta_2\eta_4 + 3\eta_2 + 3\eta_4 + 11)\Big\} f_2 f_4,
\end{aligned}
\tag{4.35}
$$

equivalent to (4.21)-(4.23), where

$$
\eta_j \equiv \frac{a}{f_j}\frac{\partial f_j}{\partial a}, \tag{4.36}
$$

and where we have abbreviated

$$\bar{\rho} = \frac{3}{a^3}\int_0^a \rho a^2 da, \quad D = \frac{\rho}{\bar{\rho}}. \tag{4.37}$$

If, in Eq. (4.33), all terms of its right-hand side are disregarded, its left-hand side equated to zero will reduce to Clairaut's well-known equation defining a first-order distortion of the rotating configuration. If, in Eqs. (4.33) and (4.34), we suppress third-order terms factored by f_2^3 or f_2f_4, we obtain the second-order generalizations of Clairaut's equations which were previously deduced by Kopal (6, Sec. III-5); but their generalization to third-order terms, as well as Eq. (4.35) for f_6, are new. The only previous attempt to specify terms of third order by a different method than that employed by us here was by Lanzano (7); but it did not reach the stage of explicit equations presented in this report.

The boundary conditions necessary for complete specification of the particular solutions of the foregoing equations, which are to represent the amplitudes $f_j(a)$ of the individual harmonic terms on the r.h.s. of the expansion (4.1) for r', are imposed partly at the center and partly at the boundary of our configuration. As, at the center, all the $f_j(a)$'s are to be a minimum, the necessary condition for this to be so is that, for $a = 0$,

$$f_j'(0) = 0 \quad \text{for } j = 2, 4, 6, \ldots. \tag{4.38}$$

On the other hand, at the boundary $a = a_1$, all $E_j(a_1)$'s as defined by Eqs. (4.13)-(4.16) are equal to zero and for $j > 0$, the F_j's continue to be given by (4.30)-(4.32); whereas, for $j = 0$, $4\pi F_0(a_1) = m_1$ by (4.17), where m_1 denotes the total mass of our configuration. Inserting them in Eqs. (4.27)-(4.29), we find that, for $a = a_1$,

$$\begin{aligned} 2f_2 + af_2' + \frac{5}{3}\left(\frac{\omega^2 a_1^3}{Gm_1}\right) = \frac{2}{3}\left(\frac{\omega^2 a_1^3}{Gm_1}\right)&\left\{(\eta_2+5)f_2 - \frac{1}{7}(2\eta_2^2+6\eta_2+15)f_2^2\right\} \\ &+ \frac{2}{35}\left\{5(\eta_2^2+3\eta_2+6)f_2^2 - (4\eta_2^3+30\eta_2^2+60\eta_2+76)f_2^3\right. \\ &\left.+ 5(2\eta_2\eta_4+3\eta_2+3\eta_4+26)f_2f_4\right\}, \end{aligned} \tag{4.39}$$

$$\begin{aligned} 4f_4 + af_4' = \frac{2}{3}\left(\frac{\omega^2 a_1^3}{Gm_1}\right)&\left\{(\eta_4+7)f_4 - \frac{9}{35}(2\eta_2^2+10\eta_2+21)f_2^2\right\} \\ &+ \frac{18}{35}(\eta_2^2+5\eta_2+6)f_2^2 \\ &- \frac{36}{385}(3\eta_2^3+18\eta_2^2+44\eta_2+54)f_2^3 \\ &+ \frac{20}{77}(2\eta_2\eta_4+5\eta_2+5\eta_4+26)f_2f_4, \end{aligned} \tag{4.40}$$

and

$$6f_6 + af_6' = -\frac{18}{77}(\eta_2+2)(\eta_2^2+6\eta_2+12)f_2^3 + \frac{5}{11}(2\eta_2\eta_4+7\eta_2+7\eta_4+26)f_2f_4. \tag{4.41}$$

A construction (numerically or otherwise) of the desired particular solutions of the simultaneous system of differential equations (4.33)-(4.35) specified by the boundary conditions (4.38) and (4.39) (4.41) can be accomplished by successive approximations in the following manner. Within the scheme of a first-order approximation, (4.33) reduces to the well-known Clairaut equation

$$a^2 f_2'' + 6D(af_2' + f_2) = 6f_2, \tag{4.42}$$

which for

$$D = 1 - \lambda a^2 + \ldots \tag{4.43}$$

admits, in the proximity of the origin, of a solution varying as

$$f_2 = k_2\left(1 + \frac{3}{7}\lambda a^2 + \ldots\right), \tag{4.44}$$

where k_2 stands for an arbitrary constant. Integrating Eq. (4.42) can proceed hereafter until $a = a_1$, at which point the l.h.s. of Eq. (4.39) discloses that

$$2f_2 + af_2' + \frac{5}{3}\left(\frac{\omega^2 a_1^3}{Gm_1}\right) = 0; \tag{4.45}$$

and this (algebraic) equation can be used to specify the value of $\omega^2 a_1^3/Gm_1$ corresponding to the initially adopted value of k_2.

With a first-order approximation to $f_2(a)$ in our hands, we can now proceed to evaluate the r.h.s. of Eqs. (4.33) and (4.34) correctly to quantities of second order (i.e., by ignoring quantities of the order of f_2^3 or $f_2 f_4$ on their right-hand sides). Near the origin, the desired solution of Eq. (4.34) for f_4 should vary as

$$f_4 = k_4 a^2 + \ldots, \tag{4.46}$$

where k_4 stands for another constant to be determined from Eq. (4.40) at the boundary, and harmonized with k_2 for the same value of $\omega^2 a_1^3/Gm_1$. Moreover, once the second approximation to f_2 and the first approximation to f_4 have thus been obtained, we are in a position to proceed to the next iteration for a third approximation to f_2 and a second approximation to f_4, simultaneously with a first approximation to f_6 from Eq. (4.35). The latter varies near the origin as

$$f_6 = k_6 - a^4 + \ldots, \tag{4.47}$$

with k_6 a third arbitrary constant, the numerical value of which should be specified with the aid of Eq. (4.41) for the same value of $\omega^2 a_1^3/Gm_4$. In doing so, care should be taken merely to note that—unlike (4.33) and (4.34) or their associated outer boundary conditions (4.39) and (4.40)—both sides of Eqs. (4.35) or (4.41) consist of quantities which are all of the same order of magnitude.

Equations (4.33)-(4.35) subject to the boundary conditions (4.38)-(4.41) are sufficient to specify the extent of the *distortion* of our configuration, due to axial rotation, correctly to quantities of third order in ω^2. However, to investigate the secular *stability* of such a configuration by the criteria outlined in Section II, it is still necessary to ascertain the *potential energy* of the respective configuration, as well as its *moment of inertia* about the axis of rotation; and to this task we shall address ourselves in the concluding section of this report.

V. DETERMINATION OF THE POTENTIAL ENERGY

Having specified the amplitudes $f_j(a)$ on the r.h.s. of the expansion (4.1) as particular solutions of the differential equations (4.33)-(4.35) subject to the boundary conditions (4.38) and (4.39)-(4.41), we are now in a position to proceed with the evaluation of the potential energy W of our configuration from the equation

$$W = -\frac{1}{2}\int_0^{m_1} \Omega dm, \tag{5.1}$$

where Ω denotes the potential arising from the mass and where, in accordance with Eq. (3.44),

$$\begin{aligned}\Omega &= 4\pi G \sum_{j=0}^{\infty} \left\{ \frac{r'^j E_j + r'^{-j-1} F_j}{2j+1} \right\} P_j(\cos\theta) \\ &= 4\pi G \sum_{j=0}^{\infty} \beta_j(a)\, P_j(\cos\theta),\end{aligned} \tag{5.2}$$

whereas the mass element is

$$dm = \frac{\rho}{3}\frac{\partial r'^3}{\partial a}\, da \sin\theta\, d\theta\, d\phi, \tag{5.3}$$

with $r'(a, \theta, \phi)$ as given by the expansion (4.1).

The coefficients $\beta_j(a)$ in the second expansion on the r.h.s. of Eq. (5.2) are already known to us from the preceding section, for they become identical with the l.h.s. of Eqs. (4.21)-(4.23) for $j = 2$, 4, and 6. Therefore, only that for $j = 0$ remains yet to be evaluated, and this can be done in the same manner by which those for $j > 0$ have already been established. Doing so we find that, correctly to terms of the *fourth* order in superficial distortion,

$$\begin{aligned}\beta_0(a) = E_0 &+ \frac{2}{25} a^2 E_2 \left\{ f_2 + \frac{1}{7} f_2^2 - \frac{1}{5} f_2^3 + \frac{2}{7} f_2 f_4 \right\} \\ &+ \frac{4}{81} a^4 E_4 \left\{ f_4 + \frac{27}{35} f_2^2 \right\} \\ &+ \frac{F_0}{a} \left\{ 1 + \frac{2}{5} f_2^2 - \frac{4}{105} f_2^3 + \frac{43}{175} f_2^4 + \frac{2}{9} f_4^2 - \frac{4}{35} f_2^2 f_4 \right\} \\ &+ \frac{3F_2}{25a^3} \left\{ -f_2 + \frac{4}{7} f_2^2 - \frac{78}{35} f_2^3 + \frac{8}{7} f_2 f_4 \right\} \\ &+ \frac{5F_4}{82a^5} \left\{ -f_4 + \frac{54}{35} f_2^2 \right\},\end{aligned} \tag{5.4}$$

where, correctly to quantities of fourth order,

$$E_0 = \int_a^{a_1} \rho \frac{\partial}{\partial a}\left\{a^2\left[\frac{1}{2} + f_0 + \frac{1}{2} f_0^2 + \frac{1}{10} f_2^2 + \frac{1}{18} f_2^4\right]\right\} da. \tag{5.5}$$

Since, within the scheme of this approximation, Eq. (4.12) should be augmented to

$$f_0 = -\frac{1}{5} f_2^2 - \frac{2}{105} f_2^3 - \frac{1}{9} f_4^2 - \frac{2}{35} f_2^2 f_4 + \ldots \tag{5.6}$$

to safeguard the constancy of mass, it follows on its insertion in Eq. (5.5) that

$$E_0 = \int_a^{a_1} \rho \frac{\partial}{\partial a}\left\{a^2\left[\frac{1}{2} - \frac{1}{10} f_2^2 - \frac{2}{105} f_2^3 + \frac{1}{50} f_2^4 - \frac{1}{18} f_4^2 - \frac{2}{35} f_2^2 f_4\right]\right\} da \tag{5.7}$$

as a generalization of (4.13). The function F_0 continues to be given by Eq. (4.17) exactly, whereas for $j > 0$ the E_j's and F_j's are given by (4.14)-(4.16) and (4.18)-(4.20) as they already represent an approximation sufficient for the present purpose.

Moreover, to the requisite approximation,

$$r'^3 = a^3\left\{1 + 3\left[f_2 + \frac{2}{7} f_2^2 - \frac{9}{35} f_2^3 + \frac{4}{7} f_2 f_4\right] P_2 + 3\left[f_4 + \frac{18}{35} f_2^2\right] P_4 + \ldots\right\}, \tag{5.8}$$

so that if we decompose (5.1) into

$$W = W_0 + W_2 + W_4, \tag{5.9}$$

it follows from (5.3) and (5.8) that

$$W_0 = -8\pi^2 G \int_0^{a_1} \beta_0(a)\, \rho a^2 da \tag{5.10}$$

$$W_2 = -\frac{8}{5}\pi^2 G \int_0^{a_1} \beta_2(a)\left\{(3 + \eta_2) f_2 + \frac{2}{7}(3 + 2\eta_2) f_2^2 \right. \tag{5.11}$$

$$\left. - \frac{9}{35}(3 + 3\eta_2) f_2^3 + \frac{4}{7}(3 + \eta_2 + \eta_4) f_2 f_4\right\} \rho a^2 da$$

and

$$W_4 = -\frac{8}{9}\pi^2 G \int_0^{a_1} \beta_4(a)\left\{(3 + \eta_4) f_4 + \frac{18}{35}(3 + 2\eta_2) f_2^2\right\} \rho a^2 da, \tag{5.12}$$

consistently to quantities of fourth order in rotational distortion.

We note that, within this scheme of approximation, only the amplitudes f_2 and f_4 occur explicitly in the integrands on the r.h.s. of the preceding Eqs. (5.10)-(5.12), together with their logarithmic derivatives η_j but not f_6. Therefore, to evaluate the W_j's—by quadratures or otherwise—only Eqs. (4.33) and

(4.34) need to be solved for the boundary conditions (4.38)-(4.40) in terms of a suitably chosen set of generalized coordinates q_i (introduced in Section II in connection with the stability criteria for our totating configurations) which remain yet to be defined to open the way to the main objective of this report.

In order to do so, *let us identify the generalized coordinates q_i with the surface values $f_j(a_1)$ of the respective amplitudes of distortion.* In the preceding parts of this section we found it possible to express W correctly to quantities of *fourth* order in terms of *two* such coordinates, i.e.,

$$q_2 \equiv f_2(a_1) \quad \text{and} \quad q_4 \equiv f_4(a_1) \tag{5.13}$$

only. It is clear from the structure of Eqs. (5.9)-(5.12) that the potential energy W does not depend on the first powers of q_2 or q_4, but only on their squares and higher powers (or cross-products). In Section IV we succeeded in determining q_2 and q_4 correctly to quantities of third order. Therefore, q_2^2 or q_4^2 will commence to be affected by errors inherent in our scheme of approximation which are of *fifth* order in superficial distortion. Accordingly, within the scheme of fourth-order approximation, the potential energy W of our rotating configuration can be expressed as

$$\begin{aligned} W = A_0 &+ A_2 q_2^2 + A_3 q_2^3 + A_4 q_2^4 \\ &+ B_2 q_4^2 \qquad\quad + B_4 q_2^2 q_4 + \dots, \end{aligned} \tag{5.14}$$

where

$$A_0 = -8\pi^2 G \int_0^{a_1} \rho \left\{ \int_a^{a_1} \rho a\, da + \frac{1}{a} \int_0^a \rho a^2 da \right\} a^2 da \tag{5.15}$$

represents the potential energy of our configuration in its nonrotating state. For $j > 0$, the coefficients A_j and B_j can be ascertained by factorization of the individual W_j's as given by Eqs. (5.10) to (5.12).

The actual process of this factorization should indeed offer no difficulty. By introducing a nondimensional variable

$$\varphi_j(a) = \frac{f_j(a)}{f_j(a_1)} \tag{5.16}$$

constrained so that

$$0 < \varphi_j(a) \leqslant 1, \tag{5.17}$$

we can immediately rewrite Eq. (5.7) as

$$\begin{aligned} E_0 = &\int_a^{a_1} \rho a\, da - \frac{q_2^2}{10} \int_a^{a_1} \rho \frac{\partial}{\partial a}(a^2 \varphi_2^2) da - \frac{2q_2^3}{105} \int_a^{a_1} \rho \frac{\partial}{\partial a}(a^2 \varphi_2^3) da \\ &+ \frac{q_2^4}{50} \int_a^{a_1} \rho \frac{\partial}{\partial a}(a^2 \varphi_2^4) da - \frac{q_4^2}{18} \int_a^{a_1} \rho \frac{\partial}{\partial a}(a^2 \varphi_4^2) da - \frac{2q_2^2 q_4}{35} \int_a^{a_1} \rho \frac{\partial}{\partial a}(a^2 \varphi_2 \varphi_4^2) da \end{aligned} \tag{5.18}$$

and similarly for other E_j's and F_j's. The process itself is quite straightforward, but the final results for the A_j's and B_j's are too long to be printed here in full; their formulation is being left as an exercise for the interested reader.

The potential energy W represents, to be sure, only one part of the expressions (2.26) or (2.28) whose minimization should ensure the stability of our configuration. The second part involves the moment of inertia C about the axis of rotation. The explicit form of the latter can, however, be established in terms of our f_j's with equal ease: namely, from Eq. (3.2) we find that

$$C = \frac{1}{5}\int_0^{a_1} \rho \frac{\partial}{\partial a}\left\{\int_0^{\pi}\int_0^{\pi} (r')^5 \sin^3\theta \, d\theta \, d\phi\right\} da, \tag{5.19}$$

where

$$\sin^2\theta = \frac{2}{3}\left\{1 - P_2(\cos\theta)\right\} \tag{5.20}$$

and, within the scheme of our approximation,

$$\begin{aligned} r'^5 = a^5 \Bigg\{ & 1 + f_2^2 + \frac{10}{12}f_2^3 - \frac{13}{35}f_2^4 + \frac{5}{9}f_4^2 + \frac{10}{7}f_2^2 f_4 \\ & + 5\left[f_2 + \frac{4}{7}f_2^2 + \frac{2}{35}f_2^3 - \frac{184}{1155}f_2^4 + \frac{200}{693}f_4^2 \right. \\ & \left. + \frac{8}{7}f_2 f_4 + \frac{72}{77}f_2^2 f_4\right]P_2 \\ & + \text{harmonics of higher orders.}\Bigg\} \end{aligned} \tag{5.21}$$

On carrying out the requisite integrations with respect to the angular variables, we find that

$$\begin{aligned} C &= \frac{8\pi}{15}\int_0^{a_1} \rho \frac{\partial}{\partial a}\left\{a^5\left[1 - f_2 + \frac{3}{7}f_2^2 + \frac{44}{105}f_2^3 - \frac{7}{33}f_2^4 \right.\right. \\ & \qquad \left.\left. - \frac{8}{7}f_2 f_4 + \frac{185}{693}f_4^2 + \frac{38}{77}f_2^2 f_4\right]\right\} da \\ &= \frac{8\pi}{15}\int_0^{a_1} \rho a^4 \left\{5 - (5+\eta_2)f_2 + \frac{3}{7}(5+2\eta_2)f_2^2 + \frac{44}{105}(5+3\eta_2)f_2^3 \right. \\ & \quad - \frac{7}{33}(5+4\eta_2)f_2^4 + \frac{185}{693}(5+2\eta_4)f_4^2 - \frac{8}{7}(5+\eta_2+\eta_4)f_2 f_4 \\ & \quad \left. + \frac{38}{77}(5+2\eta_2+\eta_4)f_2^2 f_4\right\} da \end{aligned} \tag{5.22}$$

$$\begin{aligned} C = {} & C_0 + C_1 q_2 + C_2 q_2^2 + C_3 q_2^3 + C_4 q_2^4 \\ & + D_2 q_4^2 + D_3 q_2 q_4 + D_4 q_2^2 q_4, \end{aligned} \tag{5.23}$$

where

$$C_0 = \frac{8}{3}\pi \int_0^{a_1} \rho a^4 da, \tag{5.24}$$

$$C_1 = -\frac{8}{15}\pi \int_0^{a_1} \rho\varphi_2(5+\eta_2)a^4 da, \tag{5.25}$$

$$C_2 = \frac{24}{105}\pi \int_0^{a_1} \rho\varphi_2^2(5+2\eta_2)a^4 da, \tag{5.26}$$

$$C_3 = \frac{352}{1575}\pi \int_0^{a_1} \rho\varphi_2^3(5+3\eta_2)a^4 da, \tag{5.27}$$

$$C_4 = -\frac{56}{495}\pi \int_0^{a_1} \rho\varphi_2^4(5+4\eta_2)a^4 da; \tag{5.28}$$

and

$$D_2 = \frac{296}{2079}\pi \int_0^{a_1} \rho\varphi_4^2(5+2\eta_4)a^4 da \tag{5.29}$$

$$D_3 = -\frac{64}{105}\pi \int_0^{a_1} \rho\varphi_2\varphi_4(5+\eta_2+\eta_4)a^4 da, \tag{5.30}$$

$$D_4 = \frac{304}{1155}\pi \int_0^{a_1} \rho\varphi_2^2\varphi_4(5+2\eta_2+\eta_4)a^4 da. \tag{5.31}$$

The reader may note that, unlike the potential energy W, the moment of inertia C depends *linearly* on the generalized coordinate q_2 (though not on q_4).

In conclusion, we wish to compare the approximate results obtained above with the exact solution of the problem in the well-known case of a homogeneous spheroid of constant density. If ρ = constant and, by Eq. (4.37), $D = 1$, Eqs. (4.33)-(4.35) can obviously be satisfied with *constant* values of $f_j(a)$, so that

$$\eta_2(a) = \eta_4(a) = \eta_6(a) = 0, \tag{5.32}$$

and the boundary conditions (4.39)-(4.41) can be solved for the f_j's in the form

$$f_2 = -\frac{5}{4}\nu - \frac{25}{14}\nu^2 - \frac{23.125}{14.56}\nu^3 - \ldots, \tag{5.33}$$

$$f_4 = \quad + \frac{135}{112}\nu^2 + \frac{29025}{8624}\nu^3 + \ldots, \tag{5.34}$$

$$f_6 = -\frac{5625}{4928}\nu^3 + \ldots, \tag{5.35}$$

where

$$\nu = \frac{\omega^2}{2\pi G\rho}, \tag{5.36}$$

correctly to quantities of third order in ν.

Moreover, for constant ρ and f_j, Eqs. (5.7) together with (4.14)-(4.20) disclose that, to the requisite degree of accuracy,

$$E_0 = \rho(a_1^2 - a_2)\left(\frac{1}{2} - \frac{1}{10}f_2^2 - \frac{2}{105}f_2^3 + \frac{1}{50}f_2^4 - \frac{1}{18}f_4^2 - \frac{2}{35}f_2^2 f_4\right), \tag{5.37}$$

$$E_2 = 0, \tag{5.38}$$

$$E_4 = \rho(a_1^{-2} - a^{-2})\left(f_4 - \frac{27}{35}f_2^2\right); \tag{5.39}$$

and

$$F_0 = \frac{1}{3}\rho a^3, \tag{5.40}$$

$$F_2 = \rho a^5\left(f_2 + \frac{4}{7}f_2^2 + \frac{2}{35}f_2^3 + \frac{8}{7}f_2 f_4\right), \tag{5.41}$$

$$F_4 = \rho a^7\left(f_4 + \frac{54}{35}f_2^2\right). \tag{5.42}$$

This leads to

$$\begin{aligned}\beta_0 = {} & \rho(a_1^2 - a^2)\left(\frac{1}{2} - \frac{1}{10}f_2^2 - \frac{2}{105}f_2^3 + \frac{1}{50}f_2^4 - \frac{1}{18}\beta_4^2 - \frac{2}{35}f_2^2 f_4\right) \\ & + \frac{4}{81}\rho(a_1^{-2} - a^{-2})\left(f_4 + \frac{27}{35}f_2^2\right)\left(f_4 - \frac{27}{35}f_2^2\right) \\ & + \rho a^2\left(\frac{1}{3} + \frac{1}{75}f_2^2 - \frac{4}{315}f_2^3 - \frac{23}{3675}f_2^4 + \frac{1}{81}f_4^2 - \frac{4}{105}f_2^2 f_4\right),\end{aligned} \tag{5.43}$$

$$\begin{aligned}\beta_2 = {} & \frac{8}{63}\rho a^4(a_1^{-2} - a^{-2})\left(f_2 f_4 - \frac{27}{35}f_2^3\right) \\ & + \rho a^2\left(-\frac{2}{15}f_2 + \frac{4}{105}f_2^2 + \frac{2}{75}f_2^3 + \frac{4}{45}f_2 f_4\right),\end{aligned} \tag{5.44}$$

and

$$\begin{aligned}B_4 = {} & \frac{1}{9}\rho a^4\left(a_1^{-2} - a^{-2}\right)\left(f_4 - \frac{27}{35}f_2^2\right) \\ & - \rho a^2\left(\frac{2}{9}f_4 - \frac{6}{175}f_2^2\right).\end{aligned} \tag{5.45}$$

Accordingly, Eqs. (5.9)-(5.12) can be shown to reduce for homogeneous configurations ($\varphi_j = 1$ and $\eta_j = 0$) to

$$W = -\frac{16}{15}\pi^2 G\rho^2 a_1^5 \left\{1 - \frac{1}{5} f_2^2 - \frac{4}{105} f_2^3 + \frac{157}{1225} f_2^4 - \frac{6}{35} f_2^2 f_4 - \frac{5}{27} f_4^2\right\}; \tag{5.46}$$

And, similarly, by (5.23)-(5.31), we find that

$$C = \frac{8}{15}\pi\rho a_1^5 \left\{1 - f_2 + \frac{3}{7} f_2^2 + \frac{44}{105} f_2^3 - \frac{7}{33} f_2^4 - \frac{8}{7} f_2 f_4 + \frac{38}{77} f_2^2 f_4 + \frac{185}{693} f_4^2\right\}, \tag{5.47}$$

so that

$$\begin{aligned} I &= W - \frac{1}{2} w^2 C \\ &= -\frac{16}{15}\pi^2 G\rho^2 a_1^5 \left\{1 - \frac{1}{5} f_2^2 - \frac{4}{105} f_2^3 + \frac{157}{1225} f_2^4 \right. \\ &\quad - \frac{6}{35} f_2^2 f_4 - \frac{5}{27} f_4^2 + \frac{\nu}{2}\left[1 - f_2 + \right. \\ &\quad \left.\left. + \frac{3}{7} f_2^2 - \frac{8}{7} f_2 f_4 + \frac{44}{105} f_2^3\right]\right\}, \end{aligned} \tag{5.48}$$

correctly to terms of the fourth order in ν.

By a differentiation of I with respect to $q_2 \equiv f_2$ and $q_4 \equiv f_4$ we find that the equilibrium condition

$$\frac{\partial I}{\partial q_2} = 0 \tag{5.49}$$

is satisfied to quantities of the third order in ν, and

$$\frac{\partial I}{\partial q_4} = 0 \tag{5.50}$$

to quantities of second order—whatever the value of ν happens to be.

When we turn to the evaluation of the partial derivatives of I with respect to $q_{1,2}$ of second order, we find that

$$\frac{\partial^2 I}{\partial q_2^2} = \frac{16}{15}\pi^2 G\rho^2 a_1^5 \left\{\frac{2}{5} - \frac{5}{7}\nu - \frac{243}{4.49}\nu^2\right\}, \tag{5.51}$$

$$\frac{\partial^2 I}{\partial q_4^2} = \frac{16}{15}\pi^2 G\rho^2 a_1^5 \left\{\frac{10}{27}\right\}, \tag{5.52}$$

and

$$\frac{\partial^2 I}{\partial q_2 \partial q_4} = \frac{16}{15}\pi^2 G\rho^2 a_1^5 \left\{\frac{1}{7}\nu - \frac{30}{49}\nu^2\right\}. \tag{5.53}$$

Accordingly, the coefficients of stability $b_{1,2}$ as defined by Eqs. (2.13) and (2.14) assume the explicit form

$$b_1 = \frac{16}{15}\pi^2 G\rho^2 a_1^5 \left\{\frac{2}{5} - \frac{5}{7}\nu - \frac{243}{4.49}\nu^2\right\} \tag{5.54}$$

and

$$b_1 b_2 = \left\{\frac{16}{15}\pi^2 G\rho^2 a_1^5\right\}^2 \begin{vmatrix} \frac{2}{5} - \frac{5}{7}\nu - \frac{243}{4.49}\nu^2 & \frac{1}{7}\nu - \frac{30}{49}\nu^2 \\ \frac{1}{7}\nu - \frac{30}{49}\nu^2 & \frac{10}{27} \end{vmatrix} . \tag{5.55}$$

If terms quadratic in ν were ignored, which would be tantamount to dropping fourth-order terms in (5.48), the positivity of b_1 would require that

$$\nu < \frac{14}{25} = 0.56; \tag{5.56}$$

whereas, with the retention of the quadratic terms, the positivity of b_1 and b_2 would require that

$$\nu < 0.349 \qquad \text{for } b_1 > 0, \tag{5.57}$$

and

$$\nu < 0.345 \qquad \text{for } b_1 b_2 > 0, \tag{5.58}$$

respectively.

These approximations represent, to be sure, rather strong in equalities, for it is known (5, p. 41) that a homogeneous mass constrained to remain a figure of revolution will lose stability only when ν = 0.255, a value which diminishes further to 0.187 if the configuration is allowed to become nonspheroidal. However—and this is essential—the errors of the expansions developed in this report are maximum for homogeneous configurations and diminish rapidly with increasing degrees of central condensation. For stellar models with ratios of the central-to-mean density of the order of 100, the errors of the approximation developed here are likely not to exceed a few percent, and to become quite insensible for the models of the stars situated to the right of the main sequence in the HR-diagram. Therefore, for ordinary astrophysical purposes the approximations to the potential energy W and the moment of inertia C as developed in this investigation should generally be ample; and therein rests their present significance.

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13. ABSTRACT

This report utilizes Poincaré's criteria to investigate the secular stability of self-gravitating configurations of arbitrary structure in the state of rapid rotation. The potential energy, a knowledge of which is necessary for application of these criteria, has been determined by an extension of Clairaut's method, and its evaluation in terms of suitably chosen generalized coordinates carried out explicitly to quantities of fourth order in superficial oblateness, for configurations of arbitrary internal structure.

The method employed can, moreover, clearly be extended to attain accuracy of any order, at the expense of mere manipulative work which lends itself to machine automation. The angular velocity of axial rotation can be an arbitrary function of position as well as of the time. An application of our results to homogeneous configurations in rigid-body rotation was undertaken to demonstrate that our method, when applied to a case for which a closed solution exists, leads to results which are consistent with it.

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